NUMERICAL REPRESENTATION OF PLANETARY EPHEMERIDES

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ABSTRACT. The Jet Propulsion Laboratory provides high-precision numerically integrated planetary and lunar ephemerides in support of spacecraft navigation and other activities relating to solar system bodies. Hundreds of users around the world have requested copies of the ephemerides. In the interests of compactness and utility, techniques have been developed for (1) the generation of the coefficients of an interpolating polynomial based on output from the integrator, and (2) transformation of the contents of an ephemeris file to a standard form usable on virtually any computer.

1. Representation by Chebyshev Coefficients

1.1. THE NEED FOR INTERPOLATION

The numerical integration program carries the instantaneous state of the solar system as polynomials in the form of position, velocity, acceleration, and up to 14 modified divided backward differences of acceleration for each cartesian component of the nine planets and the Moon. Saving the difference arrays at every integration step would result in prohibitively large files. Most applications of the ephemeris require only the positions of designated bodies. Considerable economy of file size is achieved by obtaining polynomial representations of the positions valid over a certain time span.

1.2. CHEBYSHEV POLYNOMIALS

Chebyshev polynomials are the functions of choice for ephemeris representation: they are stable during evaluation, they give a near-minimax representation, and they provide a readily apparent estimate of neglected terms on interpolation error. For an extensive discussion of these polynomials, see Rivlin, 1974.

The nth Chebyshev polynomial $T_n(t)$ is defined by the recursion formula

$$T_n(t) = 2\,t\,T_{n-1}(t) - T_{n-2}(t), \qquad n = 2, 3, \ldots \tag{1}$$

with $T_0(t) = 1$ and $T_1(t) = t$. The applicable range of t for interpolation is $-1 \leq t \leq 1$.

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Any given function $f(t)$ has an approximate Nth-degree expansion in Chebyshev polynomials:

$$f(t) \doteq \sum_{n=0}^{N} a_n T_n(t), \tag{2}$$

and, when differentiated,

$$\dot{f}(t) \doteq \sum_{n=1}^{N} a_n \dot{T}_n(t), \tag{3}$$

where the a_n are chosen in a manner appropriate for $f(t)$ and $\dot{f}(t)$. In the present case, where $f(t_j)$ and $\dot{f}(t_j)$ denote a coordinate and its derivative computed at discrete times t_j by the integrating program, the a_n serve to *define* the function $f(t)$ as a polynomial. The task becomes the determination of a set of a_n that provide interpolated values suitably approximating those available from the original backward-difference representations carried by the integrator. The following section details the generation of the a_n for ephemeris body coordinates. (It should be noted that the use of Chebyshev polynomials to repesent ephemerides is not new. The Jet Propulsion Laboratory has distributed Chebyshev-constructed files since 1974; the *Connaissance des Temps* has been available as Chebyshev polynomials since 1978, in both printed and machine-readable form.)

1.3. CHEBYSHEV COEFFICIENT GENERATION

The subroutine PVCH was developed to provide efficient generation of the Chebyshev polynomial coefficients representing the cartesian coordinates of the ephemeris bodies. The full span of, say, sixty years of an ephemeris file is segmented into contiguous intervals, or granules, of fixed length. (The actual length of a granule depends on the body; see Table 1 for details.) For each coordinate of an ephemeris body the Chebyshev coefficients a_n that define the interpolating polynomial valid over a given granule must be produced. There are as many sets of coefficients representing each coordinate as there are granules covering the ephemeris span.

PVCH accepts a pair of position and velocity values from the integrator for a given granule at each of the nine (equally spaced) normalized times: $1, 3/4, 1/2, 1/4, \ldots, -3/4, -1$. The output is the set of a_n for the polynomial that is an exact fit to the position values at the end points ($t = 1$, $t = -1$) and a least-squares fit to the interior positions, and whose differentiated polynomial is an exact fit to the velocities at the end points and a least-squares fit to the interior velocities. This approach has the advantage that interpolated position and velocity are continuous at the common end point of adjacent granules; it also minimizes the effects of noise that would otherwise degrade the interpolated velocity obtained from differentiation of a polynomial based on position values alone.

PVCH uses a set of multipliers c_k to obtain the a_n as a linear combination of the input positions and velocities:

$$\begin{aligned} a_n(N) = {} & c_1(N)\,p(1) + c_2(N)\,v(1) + c_3(N)\,p(3/4) + c_4(N)\,v(3/4) \\ & + \cdots + c_{17}(N)\,p(-1) + c_{18}(N)\,v(-1) \end{aligned}$$

The $c_k(N)$ are unique for each a_n and for each polynomial degree N. In the application of the following steps the $c_k(N)$ were formed and encoded as DATA statements intended for PVCH. That subroutine is subsequently used to form the entire set of Chebyshev coefficients constituting an ephemeris file. The $c_k(N)$ were obtained as follows: from Eq. (2) there are 18 conditions on the a_n:

$$
\begin{aligned}
p(1) &= \sum_{n=0}^{N} a_n T_n(1) & v(1) &= \sum_{n=0}^{N} a_n \dot{T}_n(1) \\
p(3/4) &= \sum_{n=0}^{N} a_n T_n(3/4) & v(3/4) &= \sum_{n=0}^{N} a_n \dot{T}_n(3/4) \\
&\vdots & &\vdots \\
p(-1) &= \sum_{n=0}^{N} a_n T_n(-1) & v(-1) &= \sum_{n=0}^{N} a_n \dot{T}_n(-1)
\end{aligned}
\tag{4}
$$

This system can be expressed in matrix and vector notation as $\mathsf{T}\mathbf{a} = \mathbf{f}$, or

$$
\begin{bmatrix}
T_0(1) & T_1(1) & T_2(1) & \dots & T_N(1) \\
\dot{T}_0(1) & \dot{T}_1(1) & \dot{T}_2(1) & \dots & \dot{T}_N(1) \\
T_0(3/4) & T_1(3/4) & T_2(3/4) & \dots & T_N(3/4) \\
\dot{T}_0(3/4) & \dot{T}_1(3/4) & \dot{T}_2(3/4) & \dots & \dot{T}_N(3/4) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_0(-1) & T_1(-1) & T_2(-1) & \dots & T_N(-1) \\
\dot{T}_0(-1) & \dot{T}_1(-1) & \dot{T}_2(-1) & \dots & \dot{T}_N(-1)
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}
=
\begin{bmatrix} p(1) \\ v(1) \\ p(3/4) \\ v(3/4) \\ \vdots \\ p(-1) \\ v(-1) \end{bmatrix}
\tag{5}
$$

The matrix T is $18 \times (N+1)$, implying that the system is overdetermined for $N < 17$ and must be solved by a least-squares method. In addition the requirement that the end-point positions and velocities be reproduced exactly imposes four constraints on the system.

1.4. SOLUTION WITH LAGRANGE MULTIPLIERS

We want to minimize the Euclidean norm $\| \mathsf{T}\mathbf{a} - \mathbf{f} \|^2$, subject to the four constraints

$$
\begin{aligned}
g_1(\mathbf{a}) &= \sum_{n=0}^{N} a_n T_n(1) - p(1) = 0, & g_3(\mathbf{a}) &= \sum_{n=0}^{N} a_n T_n(-1) - p(-1) = 0, \\
g_2(\mathbf{a}) &= \sum_{n=0}^{N} a_n \dot{T}_n(1) - v(1) = 0, & g_4(\mathbf{a}) &= \sum_{n=0}^{N} a_n \dot{T}_n(-1) - v(-1) = 0.
\end{aligned}
\tag{6}
$$

Following the standard use of Lagrange multipliers, for each a_n we have

$$
\frac{\partial}{\partial a_n}\left(\| \mathsf{T}\mathbf{a} - \mathbf{f} \|^2 + \sum_{i=1}^{4} \lambda_i g_i(\mathbf{a}) \right) = 0. \tag{7}
$$

Differentiating with respect to each of the a_n separately yields $N+1$ equations in $N+5$ unknowns (the λ_i must be included). Appending the four constraint equations (6) produces the $(N+5)\times(N+5)$ system:

$$
\left[\begin{array}{cccc|cccc}
 & & & & T_0(1) & \dot{T}_0(1) & T_0(-1) & \dot{T}_0(-1) \\
 & \mathsf{T^*WT} & & & T_1(1) & \dot{T}_1(1) & T_1(-1) & \dot{T}_1(-1) \\
 & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & T_N(1) & \dot{T}_N(1) & T_N(-1) & \dot{T}_N(-1) \\
\hline
T_0(1) & T_1(1) & \ldots & T_N(1) & 0 & 0 & 0 & 0 \\
\dot{T}_0(1) & \dot{T}_1(1) & \ldots & \dot{T}_N(1) & 0 & 0 & 0 & 0 \\
T_0(-1) & T_1(-1) & \ldots & T_N(-1) & 0 & 0 & 0 & 0 \\
\dot{T}_0(-1) & \dot{T}_1(-1) & \ldots & \dot{T}_N(-1) & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}
$$

$$
= \left[\begin{array}{ccccccc}
 & & & \mathsf{T^*W} & & & \\
\hline
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} p(1) \\ v(1) \\ p(3/4) \\ v(3/4) \\ \vdots \\ p(-1) \\ v(-1) \end{bmatrix} \tag{8}
$$

The notation T^* denotes the transpose of T. The matrix W is a diagonal weighting matrix, included to allow control of the relative effects of position and velocity values. It was found experimentally that the best results are obtained with velocity weighted at .4 relative to position, giving W the form diag(1.0, 0.16, 1.0, 0.16, ...).

The above matrix equation (8) can be written as $\mathsf{C}_1\mathbf{a}_\lambda = \mathsf{C}_2\mathbf{f}$, where $\mathbf{a}_\lambda$ is the vector $\mathbf{a}$ augmented by the λ_i. The solution is $\mathbf{a}_\lambda = (\mathsf{C}_1^{-1}\mathsf{C}_2)\mathbf{f}$. Row $n+1$ of the product matrix $(\mathsf{C}_1^{-1}\mathsf{C}_2)$ contains the multipliers $c_k(N)$ for each $a_n(N)$. It is these $c_k(N)$ that are formatted for the `DATA` statements in PVCH.

The solutions for the λ_i are also part of the result. However, they have no useful interpretation in this context and are ignored.

1.5. INTERPOLATION ERROR AND POLYNOMIAL DEGREE

It is essential to have a quantitative estimate of the maximum error expected from the interpolation process when the polynomials described above are used to extract coordinate values at arbitrary times. (It should be noted that the term "error" here refers to the difference between interpolated and integrator-supplied values; it does *not* indicate the degree of accuracy to which the original integrated ephemeris represents the dynamical state of the solar system.)

The Chebyshev polynomials provide a convenient and reliable estimate of interpolation error. An arbitrary function has the exact representation as an infinite Chebyshev expansion

$$f(t) = \sum_{n=0}^{\infty} a_n T_n(t). \tag{9}$$

The maximum value of $T_n(t)$ is unity on the interval $[-1, 1]$, the domain of validity for interpolation. Therefore, when a function is approximated by an Nth-degree polynomial, as in Eqn. (2), the maximum error ϵ arising from the omitted remainder of the series has the upper bound

$$\epsilon = \left| \sum_{n=N+1}^{\infty} a_n T_n(t) \right| \leq \sum_{n=N+1}^{\infty} |a_n| \, |T_n(t)| \leq \sum_{n=N+1}^{\infty} |a_n|. \tag{10}$$

Investigation has shown that the granule length and the polynomial degree N can be chosen so that $|a_{n+1}/a_n| \approx 0.1$ or less for $n \geq N$, implying that the maximum expected interpolation error is about one tenth the magnitude of the highest retained coefficient a_N.

The accuracy criterion for standard JPL ephemerides is that the interpolation error for all coordinate values must be less than 0.5 millimeters. (The DE102 ephemeris covers 4400 years; in the interests of providing a significantly compressed file the interpolation-error criterion was relaxed by defining polynomials of a given degree that span granules of twice the length of those on other JPL ephemerides [Newhall *et al.*, 1983].) The minimum degree N of the interpolating polynomial is 3, as the requirement that the end-point position and velocity values be matched exactly yields four constraints; the 18 combined position and velocity values permit a maximum degree of 17. Table 1 lists the granule length and polynomial degree for each body on the JPL ephemeris files.

Table 1. Granule Length and Polynomial Degree for the 11 Ephemeris Bodies

Body	Granule Length (days)	Polynomial Degree N	Body	Granule Length (days)	Polynomial Degree N
Mercury	8	13	Saturn	32	6
Venus	16	9	Uranus	32	5
Earth-Moon Barycenter	16	12	Neptune	32	5
			Pluto	32	5
Mars	32	10	Moon	4	12
Jupiter	32	7	Sun	16	10

2. Exporting the Ephemeris

2.1. PORTABILITY

Because of the accuracy requirments of space navigation the JPL ephemerides are suffient to satisfy the most stringent demands of any application. Hundreds of users from all over the world, representing virtually every type of computer available, have requested a copy of the ephemerides. It is essential that a thoroughly portable, machine-independent version of the files and software be produced.

2.2. DATA REPRESENTATION

The JPL ephemeris files consist of character, integer, and double-precision floating point data types. The files are originally produced on a Univac 1100/91 computer. When transformed for an export tape the file contents are written on a tape in what amounts to a formatted dump of the file. Character data are written as `CHARACTER*6` variables; integers are written as integers.

Double-precision data present a problem. Fortran floating point printed numbers are in general not exact representations of the binary quantities. On the Univac, floating point numbers are represented as a sign bit, an 11-bit biased exponent of 2, and a 60-bit mantissa, with the binary point assumed to be at the left of the mantissa. (The exponent bias is 1024_{10}, or 2000_8.) In the export format each double-precision number is written as three integers: N_1 = the exponent k with the bias removed, followed by the mantissa expressed as two 30-bit integers N_2 and N_3. When reconstructed on the receiving machine, the formula is

$$\text{Double-precision number} = N_2 \times 2^{N_1-30} + N_3 \times 2^{N_1-60}$$

As an illustrative example, on DE200 the conversion factor between AU's and kilometers is 149597870.66. It has the octal floating-point representation

$$2034\ 4352535272\ 5075341217.$$

When converted to export format the three-integer representation becomes

$$\begin{aligned} N_1 &= 28 & &(= 34_8) \\ N_2 &= 598391482 & &(= 4352535272_8) \\ N_3 &= 687194767 & &(= 5075341217_8) \end{aligned}$$

2.3. THE EXPORT SOFTWARE

The software package on the export tape sent to users contains a program for decoding the tape and creating a direct-access file on mass storage. It also includes subroutines for subsequent reading and interpolating the ephemeris file.

3. Acknowledgment

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4. References

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